On sets of odd type of $PG(n, 4)$ and the universal embedding of the dual polar space $DH(2n-1, 4)$

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Abstract

A set of points of the projective space $PG(n, 4)$, $n \geq 0$, is said to be of odd type if it intersects each line in an odd number of points. The number of sets of odd type of $PG(n, 4)$, $n \geq 0$, is known to be equal to 2^{a_n} where $a_n = \frac{1}{3}(n+1)(n^2+2n+3)$. In the present paper, we give an alternative more geometric proof of this property. The additional information revealed by this proof will allow us to prove some facts regarding the hyperplanes and the universal embedding of the Hermitian dual polar space $DH(2n-1, 4)$.

Keywords: sets of points of projective spaces, dual polar space, hyperplane, embedding
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1 Introduction

A set of points of the projective space $PG(n, 4)$, $n \geq 0$, is said to be of *odd type* if it intersects each line in an odd number of points, i.e. in either 1, 3 or 5 points.

Let (x, Π) be a non-incident point-hyperplane of $PG(n, 4)$ and Y a set of odd type of $\Pi \cong PG(n-1, 4)$. Then the cone xY with vertex x and basis Y is a set of odd type of $PG(n, 4)$. We call any such set a *conical set of odd type* of $PG(n, 4)$.

Let $\mathcal{P}_n$ denote the point set of $PG(n, 4)$. If X_1 and X_2 are two sets of odd type of $PG(n, 4)$, then the complement $\mathcal{P}_n \setminus (X_1 \Delta X_2)$ of the symmetric difference $X_1 \Delta X_2$ of X_1 and X_2 is again a set of odd type of $PG(n, 4)$. This fact allows us to give the set $\mathcal{F}_n$ of all sets of odd type of $PG(n, 4)$ the structure of an $\mathbb{F}_2$-vector space by defining $0 \cdot X := \mathcal{P}_n$, $1 \cdot X := X$ and $X_1 + X_2 := \mathcal{P}_n \setminus (X_1 \Delta X_2)$ for all $X, X_1, X_2 \in \mathcal{F}_n$. So, the number of sets of odd type of $PG(n, 4)$ is equal to 2^{a_n}, where a_n is the dimension of the $\mathbb{F}_2$-vector space $\mathcal{F}_n$.

It is known that $a_n = \frac{1}{3}(n+1)(n^2+2n+3)$, see Sherman [10, Corollary 1]. By [10, Corollary 2] we also know that $a_n = \frac{4^{n+1}-1}{3} - b_n$, where b_n is the $\mathbb{F}_2$-rank of the incidence matrix of points and lines of $PG(n, 4)$. The ranks of incidence matrices involving subspaces of finite projective spaces have been studied by many people. Complicated formulas which enable to compute b_n for any n can be found in Hamada [3, Theorem 1] or Inamdar and Sastry [7, Theorem 2.13]. These formulas are however not so easy to work with. In fact, it seems even very hard to derive b_n from these formulas.

The techniques exposed in [10] are algebraic of nature and aim to give explicit equations for the sets of odd type involved. In the present paper we give an alternative more geometric/combinatorial proof for the fact that a_n is equal to $\frac{1}{3}(n+1)(n^2+2n+3)$. The proof also reveals some additional information which is essential to derive some properties of the hyperplanes and the universal embedding of the dual polar space $DH(2n-1,4)$, $n \geq 2$. We will prove the following in Section 2.

Theorem 1.1 (1) *We have* $a_n = \frac{1}{3}(n+1)(n^2+2n+3)$.

(2) *If* X *is a set of odd type of* $PG(n,4)$*,* $n \geq 3$*, then there exist conical sets* X_1*,* X_2 *and* X_3 *of odd type of* $PG(n,4)$ *such that* $X = X_1 + X_2 + X_3$*.*

In fact, more restrictions on the sets X_1, X_2 and X_2 occurring in Theorem 1.1(2) can be imposed. We can choose two arbitrary points p_1 and p_2 and require that X_i, $i \in \{1,2\}$, is a cone with vertex p_i and that X_3 is a cone whose vertex can be chosen to be an $(n-3)$-dimensional subspace. We refer to Section 2 for more details. We will give a proof for Theorem 1.1(1) by showing that the numbers a_n satisfy a linear difference equation of degree 2. Together with the initial values $a_0 = 1$, $a_1 = 4$ and $a_2 = 11$, this will allow us to obtain an explicit expression for a_n.

Let $H(2n-1,4)$, $n \geq 2$, denote a nonsingular Hermitian variety of $PG(2n-1,4)$. With $H(2n-1,4)$, there is associated a dual polar space $DH(2n-1,4)$. This is the point-line geometry whose points are the $(n-1)$-dimensional subspaces of $PG(2n-1,4)$ which are contained in $H(2n-1,4)$ (the so-called *generators of* $H(2n-1,4)$) and whose lines are the $(n-2)$-dimensional subspaces of $PG(2n-1,4)$ which are contained in $H(2n-1,4)$, with incidence being reverse containment. We will often regard the lines of $DH(2n-1,4)$ as sets of points. The dual polar space $DH(3,4)$ is isomorphic to the generalized quadrangle $Q(5,2)$ associated with a nonsingular elliptic quadric of $PG(5,2)$. If α is an $(n-3)$-dimensional subspace of $PG(2n-1,4)$ contained in $H(2n-1,4)$, then the set Q_α of generators of $H(2n-1,4)$ through α is a subspace of $DH(2n-1,4)$, called a *quad*. The lines of $DH(2n-1,4)$ which have all their points in Q_α give Q_α the structure of a generalized quadrangle isomorphic to $Q(5,2)$. If x is a point of $\Delta = DH(2n-1,4)$, then the lines and quads through x define a linear space $Res_\Delta(x)$ which is isomorphic to (the point-line system of) $PG(n-1,4)$. If no confusion is possible, we will also write $Res(x)$ instead of $Res_\Delta(x)$.

A *hyperplane* of $DH(2n-1,4)$ is a set of points of $DH(2n-1,4)$, distinct from the whole point set, which intersects each line of $DH(2n-1,4)$ in either a singleton or the whole line. If H is a hyperplane of $DH(2n-1,4)$ and $x \in H$, then $\Lambda_H(x)$ denotes the set of all lines through x contained in H. The following holds.

Lemma 1.2 *If H is a hyperplane of $DH(2n-1,4)$, $n \geq 2$, and $x \in H$, then $\Lambda_H(x)$ is a set of odd type of $Res(x) \cong PG(n-1,4)$.*

Proof. This was proved in De Bruyn and Pralle [2, Proposition 2.3] for the case $n = 3$ but the proof remains valid for any $n \geq 2$. The idea is as follows. A line L of $Res(x)$ corresponds to a quad $Q \cong Q(5,2)$ through x. There are three possible intersections of Q with H: (i) Q itself; (ii) a subquadrangle of Q isomorphic to $Q(4,2)$; (iii) the set of points of Q equal to or collinear with a given point of Q. We readily see that there are 1, 3 or 5 lines of Q through x which are contained in $Q \cap H$. ■

We will invoke Theorem 1.1(2) and the classification of the hyperplanes of the dual polar spaces $DH(5,4)$ (obtained in De Bruyn and Pralle [1, 2]) to prove the "converse" of Lemma 1.2.

Theorem 1.3 *Let x be a point of the dual polar space $DH(2n-1,4)$, $n \geq 2$, and A a set of odd type of $Res(x)$. Then there exists a hyperplane of $DH(2n-1,4)$ through x such that $\Lambda_H(x) = A$.*

The dual polar space $DH(2n-1,4)$, $n \geq 2$, admits at least one full projective embedding and hence also the universal embedding by Ronan [9]. This universal embedding is obtained as follows. Let V be a vector space over the field $\mathbb{F}_2$ of order 2 with a basis B whose vectors are indexed by the points of $DH(2n-1,4)$, say $B = \{\bar{v}_p \,|\, p \in \mathcal{P}\}$ with $\mathcal{P}$ the point set of $DH(2n-1,4)$. Let W denote the subspace of V generated by all vectors $\bar{v}_{p_1} + \bar{v}_{p_2} + \bar{v}_{p_3}$ where $\{p_1, p_2, p_3\}$ is a line of $DH(2n-1,4)$. Then the map $p \in \mathcal{P} \mapsto \{\bar{v}_p + W, W\}$ defines a full embedding of $DH(2n-1,4)$ into the projective space $PG(V/W)$. This embedding is precisely the universal embedding of $DH(2n-1,4)$. In this paper we will use Theorem 1.3 to prove the following.

Theorem 1.4 *Let e denote the universal embedding of the dual polar space $DH(2n-1,4)$, $n \geq 2$, into the projective space Σ. Let x be a point of $DH(2n-1,4)$ and let $x^\perp$ denote the set of points of $DH(2n-1,4)$ equal to or collinear with x. Then the (projective) dimension of the subspace $\langle e(x^\perp)\rangle_\Sigma$ of Σ is equal to $\frac{1}{3}n(n^2+2)$.*

Remarks. (a) It was the connection between sets of odd type of $PG(n,4)$ and hyperplanes of $DH(2n-1,4)$ (cf. Lemma 1.2) which stimulated the author to study these sets of points in $PG(n,4)$. It was after obtaining the results stated in Theorem 1.1 that he was informed by J.W.P. Hirschfeld about the results of the paper [10]. It is also possible to derive the result stated in Theorem 1.1(2) with the more algebraic approach discussed in [10]. We have decided to keep the more geometric/combinatorial approach. It can have some value to see different approaches to the same problem.

(b) The precise value of the vector dimension of the universal embedding e of $DH(2n-1,4)$, $n \geq 2$, was open for several years. The problem was finally solved by Li [8] who proved that the vector dimension of e is equal to $\frac{4^n+2}{3}$. The dimensions of the subspaces $\langle e(\Gamma_{\leq i}(x))\rangle$, where x is a point of $DH(2n-1,4)$ and $\Gamma_{\leq i}(x)$ denotes the set of points at distance at most i from x, are not yet fully understood. These dimensions were computed by Li [8, p. 251, Table I] for every $n \in \{2, 3, \ldots, 12\}$ and every $i \in \{0, 1, \ldots, n\}$. The dimensions which we obtained in Theorem 1.4 agree with the dimensions occurring in this table.

(c) There exists a classification (up to projective equivalence) of all sets of odd type of $PG(n, 4)$ for every $n \leq 3$. In Section 2, we give these classifications for $n \in \{0, 1, 2\}$. We will need them in the proofs of Theorems 1.1 and 1.3. The classification of all sets of odd type of $PG(3, 4)$ was obtained in Hirschfeld and Hubaut [5], based on some other results of Hirschfeld and Thas [6]. An alternative proof for the classification of all sets of odd type of $PG(3, 4)$ was given in Sherman [10].

2 Proof of Theorem 1.1

The aim of this section is to prove Theorem 1.1. As before, let $\mathcal{F}_n$, $n \geq 0$, be the set of all sets of odd type of the projective space $PG(n, 4)$. As explained in Section 1, $\mathcal{F}_n$ can be regarded as a vector space of dimension a_n over the field $\mathbb{F}_2$. We denote by $\mathcal{P}_n$ the point set of $PG(n, 4)$.

We start by giving a description of all sets of odd type of $PG(n, 4)$ where $n \leq 2$. Since there are no lines in $PG(0, 4)$, every set of points of $PG(0, 4)$ is a set of odd type. Hence, $|\mathcal{F}_0| = 2$ and $a_0 = 1$. In $PG(1, 4)$, there are three types of sets of odd type: the singletons, the subsets of size 3 of $\mathcal{P}_1$ and the set $\mathcal{P}_1$ itself. Hence, $|\mathcal{F}_1| = 5 + \binom{5}{3} + 1 = 16$ and $a_1 = 4$. Also the elements of $\mathcal{F}_2$ can easily be determined by hand. The classification of the sets of odd type of $PG(2, 4)$ is given in the following proposition.

Proposition 2.1 *Let X be a set of odd type of $PG(2, 4)$. Then X is one of the following sets of points of $PG(2, 4)$:*

(*A1*) *the whole set of points;*
(*A2*) *a line;*
(*A3*) *the union of three distinct lines through a given point;*
(*A4*) *a Baer subplane;*
(*A5*) *a nonsingular Hermitian curve;*
(*A6*) *the complement of a hyperoval;*
(*A7*) *the union of a hyperoval H and a line L which is disjoint from H.*

The result stated in Proposition 2.1 can be found at several places in the literature, like Hirschfeld [4, Theorem 19.6.2] and Hirschfeld & Hubaut [5, Theorem 4]. The discussion in [4] and [5] is based on results of Tallini Scafati who studied more general problems in her papers [11, 12, 13]. In the following table, we list the number of sets of odd type of each type. Since there are $2048 = 2^{11}$ of these sets, we have $a_2 = 11$.

A1	A2	A3	A4	A5	A6	A7	Total
1	21	210	360	280	168	1008	2048

We now turn our attention to the general case. Let x_1 and x_2 be two distinct points of $PG(n,4)$, $n \geq 1$, and let Π_i, $i \in \{1,2\}$, be a hyperplane of $PG(n,4)$ through x_i not containing x_{3-i}. We denote by $A(n)$ the total number of sets of odd type containing $\Pi_1 \cup \Pi_2$. Since $PGL(n+1,4)$ acts transitively on the unordered pairs of distinct hyperplanes of $PG(n,4)$, the number $A(n)$ only depends on n and not on the chosen hyperplanes Π_1 and Π_2.

Lemma 2.2 *For every $n \geq 2$, $a_n = 2a_{n-1} - a_{n-2} + \log_2 A(n)$.*

Proof. We define a number of sets of sets of odd type.

- We denote by $\mathcal{F}^{(i)}_{n-1}$, $i \in \{1,2\}$, the set of all sets of odd type of Π_i.
- We denote by $\mathcal{F}_{n-2}$ the set of all sets of odd type of $\Pi_1 \cap \Pi_2$.
- We denote by V_1 the set of all cones of the form x_2C with $C \in \mathcal{F}^{(1)}_{n-1}$.
- We denote by W_1 the set of all sets of odd type of $PG(n,4)$ containing Π_1.
- We denote by V_2 the set of all cones of the form x_1C with C an element of $\mathcal{F}^{(2)}_{n-1}$ containing $\Pi_1 \cap \Pi_2$.
- We denote by V_3 the set of all sets of odd type of $PG(n,4)$ containing $\Pi_1 \cup \Pi_2$.
- We denote by V_2' the set of all elements of $\mathcal{F}^{(2)}_{n-1}$ of the form x_2C with $C \in \mathcal{F}_{n-2}$.
- We denote by W_2' the set of all elements of $\mathcal{F}^{(2)}_{n-1}$ containing $\Pi_1 \cap \Pi_2$.

It is easily verified that V_1, W_1, V_2 and V_3 are subspaces of $\mathcal{F}_n$ and that V_2' and W_2' are subspaces of $\mathcal{F}^{(2)}_{n-1}$. (E.g.: If the elements C_1 and C_2 of $\mathcal{F}_n$ contain the hyperplane Π_1, then Π_1 is also contained in $C_1 + C_2$. If $C, C' \in \mathcal{F}^{(1)}_{n-1}$, then $x_2C + x_2C' = x_2(C + C')$.)

Since the map $C \mapsto x_2C$ defines a linear isomorphism between the vector spaces $\mathcal{F}^{(1)}_{n-1}$ and V_1, we have $\dim(V_1) = \dim(\mathcal{F}^{(1)}_{n-1}) = a_{n-1}$. Since the map $C \mapsto x_2C$ defines a linear isomorphism between the vector spaces $\mathcal{F}_{n-2}$ and V_2', we have $\dim(V_2') = \dim(\mathcal{F}_{n-2}) = a_{n-2}$. Since the map $C \mapsto x_1C$ defines a linear isomorphism between the vector spaces W_2' and V_2, we have $\dim(V_2) = \dim(W_2')$. Since the subspace V_3 of $\mathcal{F}_{n-1}$ contains precisely $A(n)$ elements, we have $\dim(V_3) = \log_2 A(n)$.

We prove that $\mathcal{F}_n = V_1 \oplus W_1$. If $C \in \mathcal{F}^{(1)}_{n-1}$ such that $x_2C \in W_1$, then necessarily $C = \Pi_1$ and hence $x_2C = \mathcal{P}_n$. So, $V_1 \cap W_1 = 0$. If $C \in \mathcal{F}_n$, then $C = x_2(C \cap \Pi_1) + [x_2(C \cap \Pi_1) + C]$ with $x_2(C \cap \Pi_1) \in V_1$ and $x_2(C \cap \Pi_1) + C \in W_1$. Hence, $\mathcal{F}_n = V_1 \oplus W_1$ as claimed.

Applying the previous paragraph to the subspace Π_2, we find $\mathcal{F}^{(2)}_{n-1} = V_2' \oplus W_2'$.

We prove that $W_1 = V_2 \oplus V_3$. If C is an element of $\mathcal{F}^{(2)}_{n-1}$ containing $\Pi_1 \cap \Pi_2$ such that $x_1C \in V_3$, then $C = \Pi_2$ and hence $x_1C = \mathcal{P}_n$. So, $V_2 \cap V_3 = 0$. If $C \in W_1$, then $X = x_1(C \cap \Pi_2) + [x_1(C \cap \Pi_2) + C]$ with $x_1(C \cap \Pi_2) \in V_2$ and $x_1(C \cap \Pi_2) + C \in V_3$. Hence, $W_1 = V_2 \oplus V_3$ as claimed.

Since $\mathcal{F}_n = V_1 \oplus W_1$ and $W_1 = V_2 \oplus V_3$, we have $a_n = \dim(\mathcal{F}_n) = \dim(V_1) + \dim(W_1) = \dim(V_1) + \dim(V_2) + \dim(V_3) = a_{n-1} + \dim(V_2) + \log_2 A(n)$. Since $\dim(V_2) = \dim(W_2')$, $\dim(V_2') = a_{n-2}$ and $\mathcal{F}^{(2)}_{n-1} = V_2' \oplus W_2'$, we have $\dim(V_2) = \dim(W_2') = \dim(\mathcal{F}^{(2)}_{n-1}) - \dim(V_2') = a_{n-1} - a_{n-2}$ and hence $a_n = 2a_{n-1} - a_{n-2} + \log_2 A(n)$. ■

We will now derive an explicit formula for $A(n)$. Clearly, we have $A(1) = 4$. In our derivation of the precise value of $A(n)$, we shall need the following lemma.

Lemma 2.3 *If L_1 and L_2 are two distinct lines of $PG(2,4)$, then there are precisely 12 hyperovals in $PG(2,4)$ which are disjoint from $L_1 \cup L_2$.*

Proof. We count in two different ways the number of triples (K, L, H) where K, L are two distinct lines of $PG(2,4)$ and H is a hyperoval of $PG(2,4)$ disjoint from $K \cup L$. There are 168 hyperovals in $PG(2,4)$ and for each hyperoval H, there are $6 \cdot 5$ possibilities for the ordered pair (K, L). Conversely, there are $21 \cdot 20$ possibilities for (K, L). Since $PGL(3,4)$ acts transitively on the ordered pairs of distinct lines of $PG(2,4)$, the total number of hyperovals of $PG(2,4)$ disjoint from $L_1 \cup L_2$ is equal to $\frac{168 \cdot 6 \cdot 5}{21 \cdot 20} = 12$. ■

Remark. If L_1 and L_2 are two distinct lines of $PG(2,4)$, then each set of odd type of $PG(2,4)$ containing $L_1 \cup L_2$ is either of type A1, A3 or A6. Hence, there are 1 + 3 + 12 = 16 sets of odd type of $PG(2,4)$ containing $L_1 \cup L_2$. So, $A(2) = 16$.

Lemma 2.4 *Let Π_1, Π_2 and Π_3 be three mutually distinct hyperplanes of $PG(n,4)$ through a given subspace of codimension 2. If X is a set of odd type containing $\Pi_1 \cup \Pi_2 \cup \Pi_3$, then either $X = \Pi_1 \cup \Pi_2 \cup \Pi_3$ or X coincides with the whole set of points of $PG(n,4)$.*

Proof. Suppose $X \neq \Pi_1 \cup \Pi_2 \cup \Pi_3$. Let $x \in X \setminus (\Pi_1 \cup \Pi_2 \cup \Pi_3)$, put $\Pi_4 := \langle x, \Pi_1 \cap \Pi_2 \rangle$ and let Π_5 denote the hyperplane through $\Pi_1 \cap \Pi_2$ distinct from Π_1, Π_2, Π_3 and Π_4. Every line through x not contained in Π_4 intersects $\{x\} \cup \Pi_1 \cup \Pi_2 \cup \Pi_3 \subseteq X$ in four points and hence is completely contained in X. This implies that $\Pi_1 \cup \Pi_2 \cup \Pi_3 \cup \Pi_5 \subseteq X$. If $y \in \Pi_4$ and if L is a line through y not contained in Π_4, then $L \setminus \{y\} \subseteq X$ implies that $L \subseteq X$. In particular, $y \in X$. This implies that also $\Pi_4 \subseteq X$ and hence that X coincides with the whole set of points of $PG(n,4)$. ■

Lemma 2.5 *Let Π_1 and Π_2 be two distinct hyperplanes of $\Sigma = PG(n,4)$, $n \geq 2$. If X is a set of odd type containing $\Pi_1 \cup \Pi_2$, then one of the following three cases occurs.*

(1) *X coincides with the whole set of points of Σ.*

(2) *$X = \Pi_1 \cup \Pi_2 \cup \Pi_3$ where Π_3 is a hyperplane through $\Pi_1 \cap \Pi_2$ distinct from Π_1 and Π_2.*

(3) *There exists a hyperplane α of $\Pi_1 \cap \Pi_2$ and 15 subspaces $\beta_1, \beta_2, \ldots, \beta_{15}$ of codimension 2 through α such that: (i) $X = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_{15}$; (ii) $\{\beta_1, \beta_2, \ldots, \beta_{15}\}$ defines a set of points of the quotient space $\Sigma/\alpha \cong PG(2,4)$ which is the complement of a hyperoval of Σ/α. Moreover, the hyperplane α of $\Pi_1 \cap \Pi_2$ for which this holds is unique.*

Proof. By Proposition 2.1, the lemma is true if $n = 2$. So, we may suppose that $n \geq 3$. Let Π_3, Π_4 and Π_5 denote the three remaining hyperplanes through $\Pi_1 \cap \Pi_2$. By Lemma 2.4, the lemma holds if at least one of Π_3, Π_4, Π_5 is contained in X. So, we may suppose that $\Pi_i \setminus X \neq \emptyset$ for every $i \in \{3,4,5\}$.

We prove that $|\Pi_3 \setminus X| = |\Pi_4 \setminus X| = |\Pi_5 \setminus X| = 2^{2n-3}$. Let $y \in \Pi_3 \setminus X$. There are 4^{n-1} lines through y not contained in Π_3. If L is one of these lines, then since $L \cap \Pi_1 \subseteq X$

and $L \cap \Pi_2 \subseteq X$, precisely one of $L \cap \Pi_4$, $L \cap \Pi_5$ is not contained in X. Since the 4^{n-1} lines through y not contained in Π_3 define a partition of $(\Pi_4 \cup \Pi_5) \setminus (\Pi_1 \cap \Pi_2)$, we have $|\Pi_4 \setminus X| + |\Pi_5 \setminus X| = 4^{n-1}$. In a similar way one proves that $|\Pi_3 \setminus X| + |\Pi_4 \setminus X| = |\Pi_3 \setminus X| + |\Pi_5 \setminus X| = 4^{n-1}$. These three equations are sufficient to conclude that $|\Pi_3 \setminus X| = |\Pi_4 \setminus X| = |\Pi_5 \setminus X| = 2^{2n-3}$.

We deal with the case $n = 3$ separately.

Suppose $n = 3$. Then $\Pi_1 \cap \Pi_2$ is a line. Let $i \in \{3, 4, 5\}$. Since $|\Pi_i \setminus X| = 8$, we have $|\Pi_i \cap X| = 13$. By Proposition 2.1, $\Pi_i \cap X$ is the union of three distinct lines of Π_i through a certain point p_i of $\Pi_1 \cap \Pi_2$.

We prove that $p_3 = p_4 = p_5$. If this was not the case, then one of these points, say p_3, is distinct from the other two. Let K be a line of Π_3 through p_3 not contained in X and let α be an arbitrary plane through K distinct from $\langle K, \Pi_1 \cap \Pi_2 \rangle$. Then $\alpha \cap \Pi_1 \subseteq X$, $\alpha \cap \Pi_2 \subseteq X$, $\alpha \cap \Pi_3 \cap X = \{p_3\}$ and $|\alpha \cap \Pi_4 \cap X| = |\alpha \cap \Pi_5 \cap X| = 3$. It follows that $|\alpha \cap X| = 13$ and hence that $\alpha \cap X$ is the union of three distinct lines through a certain point p. Since $\alpha \cap \Pi_1 \subseteq X$ and $\alpha \cap \Pi_2 \subseteq X$, we must have $p = p_3$, but this is in contradiction with the facts that $\alpha \cap \Pi_3 \cap X = \{p_3\}$ and $|\alpha \cap \Pi_4 \cap X| = |\alpha \cap \Pi_5 \cap X| = 3$.

Now, put $p^* := p_3 = p_4 = p_5$. By the discussion above, we must have that $X = \bigcup_{L \in \mathcal{L}} L$ where $\mathcal{L}$ is some set consisting of 15 lines through p^*. So, if Π is a plane of $PG(3, 4)$ not containing p^*, then $C := \Pi \cap X$ is a set of 15 points of Π and $X = p^*C$. The set C is a set of odd type of Π. By Proposition 2.1, C must be the complement of a hyperoval of Π. This proves that the lemma is valid in the case $n = 3$.

Suppose now that $n \geq 4$. For every point x of $(X \cap (\Pi_3 \cup \Pi_4 \cup \Pi_5)) \setminus (\Pi_1 \cap \Pi_2)$, let $T(x)$ denote the set of all points $y \in \Pi_1 \cap \Pi_2$ for which $xy \subseteq X$. We have $T(x) \neq \Pi_1 \cap \Pi_2$ since otherwise $\langle x, \Pi_1 \cap \Pi_2 \rangle \subseteq X$ and this would contradict the fact that $\Pi_i \setminus X \neq \emptyset$ for every $i \in \{3, 4, 5\}$.

Let $i \in \{3, 4, 5\}$ and $x \in (\Pi_i \cap X) \setminus (\Pi_1 \cap \Pi_2)$. We prove that $T(x)$ is a hyperplane of $\Pi_1 \cap \Pi_2$. Since $T(x) \neq \Pi_1 \cap \Pi_2$, it suffices to prove that for every line L of $\Pi_1 \cap \Pi_2$, the intersection $L \cap T(x)$ is either L or a singleton. To that end, consider an arbitrary 3-space γ through $\langle L, x \rangle$ which is not contained in Π_i. Then $\gamma \cap X$ is a set of odd type of γ which contains the planes $\gamma \cap \Pi_1$ and $\gamma \cap \Pi_2$ through the line L. Since the lemma is valid for the projective space $PG(3, 4)$, we know that there are 3 possibilities for $\gamma \cap X$. For each of these possibilities, the set of points y of L for which $xy \subseteq X$ is either L or a singleton of L, i.e., $L \cap T(x)$ is either L or a singleton. This is precisely what we needed to prove.

Let $i_1, i_2 \in \{3, 4, 5\}$ with $i_1 \neq i_2$, $x_1 \in (\Pi_{i_1} \cap X) \setminus (\Pi_1 \cap \Pi_2)$ and $x_2 \in (\Pi_{i_2} \cap X) \setminus (\Pi_1 \cap \Pi_2)$. We prove that $T(x_1) = T(x_2)$. Suppose that this is not the case. Then there exists a line L in $\Pi_1 \cap \Pi_2$ disjoint from $T(x_1) \cap T(x_2)$. Put $\{y_1\} = T(x_1) \cap L$ and $\{y_2\} = T(x_2) \cap L$. Let γ be the 3-space $\langle L, x_1, x_2 \rangle$. Then $\gamma \cap X$ is a set of odd type of γ which contains the planes $\gamma \cap \Pi_1$ and $\gamma \cap \Pi_2$ through the line L. Since the lemma is valid for the projective space $PG(3, 4)$, we know that there are two possibilities for $\gamma \cap X$ (observe that case (2) cannot occur). For each of these possibilities, the following must hold: if y is a point of L for which $x_1y \subseteq X$, then also $x_2y \subseteq X$. A contradiction now follows from the fact that

$x_1y_1 \subseteq X$ and $x_2y_1 \not\subseteq X$.

Since $|\Pi_3 \setminus X| = |\Pi_4 \setminus X| = |\Pi_5 \setminus X| = 2^{2n-3}$, we have $(\Pi_3 \cap X) \setminus (\Pi_1 \cap \Pi_2) \neq \emptyset$, $(\Pi_4 \cap X) \setminus (\Pi_1 \cap \Pi_2) \neq \emptyset$ and $(\Pi_5 \cap X) \setminus (\Pi_1 \cap \Pi_2) \neq \emptyset$. By the previous paragraph we then know that there exists a hyperplane α of $\Pi_1 \cap \Pi_2$ such that $\alpha = T(x)$ for every $x \in (X \cap (\Pi_3 \cup \Pi_4 \cup \Pi_5)) \setminus (\Pi_1 \cap \Pi_2)$. We also know that there exists a set B of subspaces of codimension 2 through α such that $X = \bigcup_{\beta \in B} \beta$. Since $|\Pi_3 \setminus X| = |\Pi_4 \setminus X| = |\Pi_5 \setminus X| = 2^{2n-3}$, there are precisely two elements of $B \setminus \{\Pi_1 \cap \Pi_2\}$ contained in each Π_i, $i \in \{3,4,5\}$. It follows that $|B| = 15$. Now, if Π is a plane of $PG(n,4)$ disjoint from α, then $C := \Pi \cap X$ is a set of 15 points of Π and X is the cone αC. The set C is a set of odd type of Π. By Proposition 2.1, C must be the complement of a hyperoval of Π. So, B defines a set of points of the quotient space $\Sigma/\alpha \cong PG(2,4)$ which is the complement of a hyperoval of Σ/α. ■

Corollary 2.6 *We have $A(n) = 2^{2n}$ for every $n \in \mathbb{N} \setminus \{0\}$.*

Proof. Since $A(1) = 4$, we may suppose that $n \geq 2$. Let Π_1 and Π_2 be two given distinct hyperplanes of $PG(n,4)$. If X is a set of odd type containing $\Pi_1 \cup \Pi_2$, then one of the three cases of Lemma 2.5 occurs. There is only one possibility for X if case (1) occurs. There are three possibilities for X if case (2) occurs, corresponding to the three possibilities for Π_3. There are $\frac{4^{n-1}-1}{3} \cdot 12$ possibilities for X if case (3) occurs. For, there are $\frac{4^{n-1}-1}{3}$ possibilities for α and for given α, there are 12 possibilities for $\{\beta_1, \beta_2, \ldots, \beta_{15}\}$ by Lemma 2.3. So, we have $A(n) = 1 + 3 + \frac{4^{n-1}-1}{3} \cdot 12 = 2^{2n}$. ■

By Lemma 2.2 and Corollary 2.6, we have $a_n = 2a_{n-1} - a_{n-2} + 2n$ for every $n \geq 2$. We also know that $a_0 = 1$, $a_1 = 4$ and $a_2 = 11$. So, the numbers a_n satisfy a difference equation of degree 2. There are standard methods for solving such equations. We find that $a_n = \frac{1}{3}(n+1)(n^2+2n+3)$ for every $n \in \mathbb{N}$. This proves Theorem 1.1(1).

In the proof of Lemma 2.2, we showed that $\mathcal{F}_n = V_1 \oplus W_1$ and $W_1 = V_2 \oplus V_3$. Now, every element of V_1 is a cone with vertex x_2, every element of V_2 is a cone with vertex x_1 and by Lemma 2.5 we know that every element of V_3 is a cone whose vertex can be chosen to be an $(n-3)$-dimensional subspace of $PG(n,4)$. Theorem 1.1(2) is obvious now.

3 Proof of Theorem 1.3

Let x be a point of the dual polar space $\Delta := DH(2n-1,4)$, $n \geq 2$, and let A be a set of odd type of $Res_\Delta(x) \cong PG(n-1,4)$. We prove by induction on n that there exists a hyperplane H of $DH(2n-1,4)$ through x for which $\Lambda_H(x) = A$.

Suppose first that $n = 2$. If A is a singleton $\{L\}$ and y is a point of $L \setminus \{x\}$, then $H := y^\perp$ is a hyperplane of $DH(3,4) \cong Q(5,2)$ satisfying the required property. If $A = \{L_1, L_2, L_3\}$ is a set of size 3, then there exists a $Q(4,2)$-subquadrangle σ of $DH(3,4) \cong Q(5,2)$ containing the lines L_1, L_2, L_3 and $H := \sigma$ satisfies the required property. If A

consists of all lines of $DH(3,4) \cong Q(5,2)$ through x, then $H := x^{\perp}$ satisfies the required property.

Suppose next that $n = 3$. The hyperplanes of the dual polar space $DH(5,4)$ were classified by De Bruyn and Pralle [1, 2]. There are 9 isomorphism classes of hyperplanes which can be divided into two families. There are 5 isomorphism classes of hyperplanes arising from the so-called Grassmann embedding of $DH(5,4)$. The hyperplanes of the other 4 isomorphism classes are called the *exceptional hyperplanes* of $DH(5,4)$. By Proposition 2.1, there are seven possibilities for A. We deal separately with each of these possibilities.

(i) Suppose A consists of all points of $Res_{\Delta}(x) \cong PG(2,4)$. Then the hyperplane $H := \Gamma_{\leq 2}(x)$ of $DH(5,4)$ satisfies the required property. The hyperplane H is called the *singular hyperplane* of $DH(5,4)$ *with deepest point* x.

(ii) Suppose A is a line of $Res_{\Delta}(x) \cong PG(2,4)$. Let Q be the quad through x corresponding to A. If $y \in Q \cap \Gamma_2(x)$, then the singular hyperplane $H := \Gamma_{\leq 2}(y)$ of $DH(5,4)$ satisfies the required property.

(iii) Suppose A is the union of three lines of $Res_{\Delta}(x)$ through a common point. Let Q_1, Q_2 and Q_3 be the three quads through x corresponding to these lines. Then $Q_1 \cap Q_2 \cap Q_3$ is a line L through x. Let $y \in L \setminus \{x\}$ and let R be a quad through y not containing L. Put $M_i := Q_i \cap R$, $i \in \{1,2,3\}$, and let σ be a $Q(4,2)$-subquadrangle of $R \cong Q(5,2)$ containing the lines M_1, M_2 and M_3. If H is the set consisting of all points of R together with all points z not contained in R which are collinear with some point of σ, then H is a hyperplane of $DH(5,4)$ satisfying the required property.

(iv) Suppose A is a nonsingular Hermitian curve of $Res_{\Delta}(x) \cong PG(2,4)$. Then by De Bruyn and Pralle [2], there exists a nonexceptional hyperplane H for which $\Lambda_H(x) = A$. (Using the terminology of [2], H is a hyperplane of type III, IV or V.)

(v) Suppose A is the complement of a hyperoval of $Res_{\Delta}(x) \cong PG(2,4)$. Then by De Bruyn and Pralle [1, Section 3.2], there exists an exceptional hyperplane H on 567 points for which $\Lambda_H(x) = A$. The hyperplane H is a so-called locally subquadrangular hyperplane of $DH(5,4)$.

(vi) Suppose A is a Baer subplane of $Res_{\Delta}(x) \cong PG(2,4)$. Then by De Bruyn and Pralle [1] (especially, see Lemmas 2 and 3 and Section 3.3), there exists an exceptional hyperplane H on 407 points for which $\Lambda_H(x) = A$.

(vii) Suppose A is the union of a hyperoval of $Res_{\Delta}(x) \cong PG(2,4)$ and a line of $Res_{\Delta}(x)$ disjoint from that hyperoval. Then by De Bruyn and Pralle [1] (especially, see Lemma 5 and Section 3.4), there exists an exceptional hyperplane H on 439 points for which $\Lambda_H(x) = A$.

Suppose now that $n \geq 4$. Before giving the proof we need to recall some facts regarding maxes of $DH(2n-1,4)$ and a construction which allows to create a new hyperplane of $DH(2n-1,4)$ from two other hyperplanes.

If p is a point of $H(2n-1,4)$, then the set of generators of $H(2n-1,4)$ through p is a subspace of $DH(2n-1,4)$, called a *max*. The lines of $DH(2n-1,4)$ which are contained in a given max M of $DH(2n-1,4)$ turn M into a point-line geometry $\widetilde{M}$ which is isomorphic to $DH(2n-3,4)$. If $y \in M$, then the set of lines through y contained in M

is a hyperplane of $Res_\Delta(y)$. Conversely, every hyperplane of $Res_\Delta(y)$ arises from a unique max through y. If Q is a quad of $DH(2n-1,4)$ and M a max of $DH(2n-1,4)$, then either $Q \cap M = \emptyset$, $Q \subseteq M$ or $Q \cap M$ is a line. If y is a point of $DH(2n-1,4)$ not contained in a max M of $DH(2n-1,4)$, then y is collinear with a unique point $\pi_M(y) \in M$. If M_1 and M_2 are two disjoint maxes of $DH(2n-1,4)$, then the map $M_1 \to M_2; y \to \pi_{M_2}(y)$ is an isomorphism between the dual polar spaces $\widetilde{M_1}$ and $\widetilde{M_2}$.

If H_1 and H_2 are two distinct hyperplanes of $DH(2n-1,4)$, then the complement $H_1 * H_2$ of the symmetric difference of H_1 and H_2 is again a hyperplane of $DH(2n-1,4)$.

Lemma 3.1 *If H_1 and H_2 are two distinct hyperplanes of $DH(2n-1,4)$ through the point x, then $\Lambda_{H_1 * H_2}(x) = \Lambda_{H_1}(x) + \Lambda_{H_2}(x)$.*

Proof. Observe that $x \in H_1 * H_2$. A line L through x is contained in $H_1 * H_2$ if and only if it is contained in either 0 or 2 of the hyperplanes H_1 and H_2. ■

We now finish the proof of Lemma 3.1. In view of Theorem 1.1(2) and Lemma 3.1, it is sufficient to consider the case that A is a conical set of $Res_\Delta(x)$. So, there exists a line L through x and distinct quads $Q_1, Q_2, \ldots, Q_k$ through L such that A consists of all lines through x which are contained in at least one of the quads $Q_1, Q_2, \ldots, Q_k$. Let y be a point of $L \setminus \{x\}$, let M_x be a max through x not containing L and let M_y be a max through y not containing L. Then M_x and M_y are disjoint. Put $K_i := M_x \cap Q_i$ and $L_i := M_y \cap Q_i$ for every $i \in \{1, 2, \ldots, k\}$. Then $L_i = \pi_{M_y}(K_i)$ for every $i \in \{1, 2, \ldots, k\}$. Since M_x defines a hyperplane of $Res_\Delta(x)$, the set $\{K_1, K_2, \ldots, K_k\}$ is a set of odd type of $Res_\Delta(x)$ and a set of odd type of $Res_{\widetilde{M_x}}(x)$. Since the map $M_x \to M_y; z \to \pi_{M_y}(z)$ is an isomorphism between the dual polar spaces $\widetilde{M_x}$ and $\widetilde{M_y}$, the set $\{L_1, L_2, \ldots, L_k\} = \{\pi_{M_y}(K_1), \pi_{M_y}(K_2), \ldots, \pi_{M_y}(K_k)\}$ is a set of odd type of $Res_{\widetilde{M_y}}(y)$. By the induction hypothesis, there exists a hyperplane G of $\widetilde{M_y}$ through y such that $\Lambda_G(y) = \{L_1, L_2, \ldots, L_k\}$. Now, let H denote the set of points of $DH(2n-1,4)$ which consists of all points of M_y together with all points z not contained in M_y for which $\pi_{M_y}(z) \in G$. Then H is a hyperplane of $DH(2n-1,4)$. Clearly, $L \subseteq H$. A line U through x distinct from L is contained in H if and only if $\pi_{M_y}(U) \subseteq G$. i.e. if and only if $\pi_{M_y}(U) \in \{L_1, L_2, \ldots, L_k\}$. This precisely happens when U is contained in one of the quads $Q_1, Q_2, \ldots, Q_k$. So, we have that $\Lambda_H(x) = A$ as we needed to prove.

4 Proof of Theorem 1.4

We now prove Theorem 1.4. Let x be an arbitrary point of $\Delta = DH(2n-1,4)$, $n \geq 2$, and let $e : \Delta \to \Sigma$ denote the universal embedding of Δ. Let α be a hyperplane of $\langle e(x^\perp) \rangle_\Sigma$ not containing $e(x)$. We prove that there exists a bijective correspondence θ between the subspaces of codimension at most 1 of α and the sets of odd type of $Res_\Delta(x) \cong PG(n-1,4)$.

Before we construct this bijection θ, we need to observe a few facts. If γ is a hyperplane of Σ, then the set of all points of Δ which are mapped by e into the hyperplane γ

is a hyperplane H_γ of Δ. Conversely, if H is a hyperplane of Δ, then by Ronan [9, Corollary 2], there exists a hyperplane γ of Σ such that $H = H_\gamma$. This hyperplane γ is uniquely determined by H. For, if γ_1 and γ_2 were two distinct hyperplanes of Σ such that $H = H_{\gamma_1} = H_{\gamma_2}$, then with γ_3 denoting the unique hyperplane of Σ through $\gamma_1 \cap \gamma_2$ distinct from γ_1 and γ_2, we would have $H_{\gamma_3} = H_{\gamma_1} * H_{\gamma_2}$. But this is impossible: since $H_{\gamma_1} = H_{\gamma_2}$, $H_{\gamma_1} * H_{\gamma_2}$ must be the complete point set of Δ. Observe also that since $\Gamma_{\leq n-1}(x)$ is a hyperplane of Δ (the so-called singular hyperplane of Δ with deepest point x), there exists a hyperplane of Σ containing $e(\Gamma_{\leq n-1}(x))$. Hence, $\langle e(x^\perp)\rangle_\Sigma$ must be a proper subspace of Σ.

Now, let β be a subspace of codimension at most 1 of α. Then there exists a hyperplane γ of Σ intersecting $\langle e(x^\perp)\rangle_\Sigma$ in the subspace $\langle e(x), \beta\rangle_\Sigma$. We define $\theta(\beta) := \Lambda_{H_\gamma}(x)$. Then $\theta(\beta)$ consists of all lines L through x for which $e(L)$ is contained in $\langle e(x), \beta\rangle_\Sigma$. Hence, $\theta(\beta)$ is independent from the chosen hyperplane γ of Σ intersecting $\langle e(x^\perp)\rangle_\Sigma$ in $\langle e(x), \beta\rangle_\Sigma$.

We prove that θ is a surjective map. So, let A be an arbitrary set of odd type of $Res_\Delta(x)$. Then by Theorem 1.3, there exists a hyperplane H of Δ through x such that $\Lambda_H(x) = A$. Let γ be the unique hyperplane of Σ such that $H = H_\gamma$ and put $\beta := \alpha \cap \gamma$. Then $A = \Lambda_H(x) = \Lambda_{H_\gamma}(x) = \theta(\beta)$.

We prove that θ is an injective map. Suppose β_1 and β_2 are two distinct subspaces of codimension at most 1 of α such that $\theta(\beta_1) = \theta(\beta_2)$. Let γ_i, $i \in \{1, 2\}$, be a hyperplane of Σ such that $\gamma_i \cap \langle e(x^\perp)\rangle_\Sigma = \langle \beta_i, e(x)\rangle_\Sigma$. Since $\beta_1 \neq \beta_2$, we also have $\gamma_1 \neq \gamma_2$. Let γ_3 denote the unique hyperplane of Σ through $\gamma_1 \cap \gamma_2$ distinct from γ_1 and γ_2. Then $H_{\gamma_3} = H_{\gamma_1} * H_{\gamma_2}$. Hence, $\Lambda_{H_{\gamma_3}}(x) = \Lambda_{H_{\gamma_1}}(x) + \Lambda_{H_{\gamma_2}}(x) = \theta(\beta_1) + \theta(\beta_2) = Res_\Delta(x)$. It follows that $x^\perp \subseteq H_{\gamma_3}$, i.e. $\langle e(x^\perp)\rangle_\Sigma \subseteq \gamma_3$. So, we would have $\gamma_1 \cap \langle e(x^\perp)\rangle_\Sigma = \gamma_2 \cap \langle e(x^\perp)\rangle_\Sigma$, i.e. $\beta_1 = \beta_2$. This is in contradiction with our assumption that $\beta_1 \neq \beta_2$.

So, θ defines a bijective correspondence between subspaces of codimension at most 1 of α and the sets of odd type of $Res_\Delta(x) \cong PG(n-1, 4)$. Since there are $2^{a_{n-1}}$ sets of odd type of $PG(n-1, 4)$, the dimension of α is equal to $a_{n-1} - 1$. Hence, the dimension of $\langle e(x^\perp)\rangle_\Sigma$ is equal to $a_{n-1} = \frac{1}{3}n(n^2 + 2)$.

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